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C Adam[1,2], J Sánchez-Guillén[1,2] and A Wereszczyński[3]

[1] Departamento de Física de Partículas, Facultad de Física, Universidad de Santiago,
E-15782 Santiago de Compostela, Spain
[2] Instituto Galego de Fisica de Altas Enerxias (IGFAE), E-15782 Santiago de Compostela, Spain
[3] Institute of Physics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

E-mail: adam@fpaxp1.usc.es, joaquin@fpaxp1.usc.es and wereszczynski@th.if.uj.edu.pl

Abstract

With the help of the Cho–Faddeev–Niemi–Shabanov decomposition of the
$SU(2)$ Yang–Mills field, we find an integrable subsystem of $SU(2)$ Yang–
Mills theory coupled to the dilaton. Here integrability means the existence of
infinitely many symmetries and infinitely many conserved currents. Further, we
construct infinitely many static solutions of this integrable subsystem. These
solutions can be identified with certain limiting solutions of the full system,
which have been found previously in the context of numerical investigations
of the Yang–Mills dilaton theory. In addition, we derive a Bogomolny bound
for the integrable subsystem and show that our static solutions are, in fact,
Bogomolny solutions. This explains the linear growth of their energies with
the topological charge, which has been observed previously. Finally, we discuss
some generalizations.

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1. Introduction

Pure Yang–Mills theory does not allow for static finite energy solutions, as follows from the
scale invariance of this theory. One simple way to circumvent this obstacle is to couple the
Yang–Mills Lagrangian to the dilaton field [1, 2]. The corresponding Lagrangian reads

$$\mathcal{L} = \tfrac{1}{4}\left(2\partial^\mu\xi\,\partial_\mu\xi - \mathrm{e}^{-2\kappa\xi}F^{a\mu\nu}F^a_{\mu\nu}\right) \tag{1}$$

where $F^a_{\mu\nu}$ is the $SU(2)$ Yang–Mills field strength, and we choose units such that the gauge
field coupling is equal to one. Further, ξ is the dilaton field and κ is the dilaton coupling
constant. A direct physical application of the above theory would require the inclusion of
further interactions (e.g., the coupling to gravity for a low-energy effective theory of string
theory), but in this paper we deal with the theory given by (1) for the sake of simplicity.

Static solutions of the Yang–Mills dilaton theory were first discussed in [1, 2], and
in these papers infinitely many unstable, sphaleron-type solutions were found numerically

within a spherically symmetric ansatz. Further, in [2] an effectively abelian solution within the spherically symmetric ansatz was found analytically, which provided a limiting case for the numerical solutions. A similar analysis was performed in [3], this time for an ansatz with only cylindrical symmetry. Again, sequences of infinitely many sphaleron solutions, labeled by a winding number m, were found numerically, and infinitely many effectively abelian limiting solutions characterized by the same winding number were constructed analytically. Further, it was observed that the energies of the limiting solutions grow linearly with the winding number m.

We shall find that all these effectively abelian limiting solutions belong to an integrable submodel of Yang–Mills dilaton theory characterized by infinitely many symmetries and infinitely many conserved currents, and that in this submodel there exists a Bogomolny bound, explaining the linear growth of energy with winding number. Recently, further solutions with only discrete symmetries (rotational symmetries of platonic bodies) have been investigated numerically in [4].

Our paper is organized as follows. In section 2, we briefly review the Cho–Faddeev–Niemi–Shabanov (CFNS) decomposition of the $SU(2)$ Yang–Mills field. Then we use this decomposition to define a submodel of Yang–Mills dilaton theory by restricting the decomposition fields, and show that this submodel is integrable in the sense that it has infinitely many symmetries and infinitely many conserved currents. In section 3 we use a separation of variable ansatz for static configurations of the integrable submodel and show that the ordinary differential equations (ODEs) obtained in this way can be solved by quadratures. The resulting solutions are precisely the limiting solutions of [2, 3] with spherical and cylindrical symmetry, respectively. In section 4 we show that there exists a Bogomolny bound for the integrable submodel, and demonstrate that the solutions of section 3 are, in fact, Bogomolny solutions. In section 5 we discuss some generalizations allowing for additional solutions of the integrable submodel, and introduce some more general integrable Lagrangians, which are no longer related to the Yang–Mills dilaton theory, but still have static field equations which can be solved by quadratures. Section 6 contains our conclusions. In the appendix, we prove that the two possible ways to derive the field equations of the integrable subsystem (restriction to the subsystem already in the Lagrangian, on the one hand, or insertion of the restriction into the field equations of the full Yang–Mills dilaton theory, on the other hand) really lead to the same field equations.

2. The integrable subsystem

The Cho–Faddeev–Niemi–Shabanov (=CFNS) decomposition (see, e.g., [5–10]) expresses the gauge field as a sum of three terms as follows:

$$A_\mu^a = n^a C_\mu + \epsilon^{abc} n_\mu^b n^c + W_\mu^a, \tag{2}$$

where $n^a = (n^1, n^2, n^3)$ is a unit vector in the $SU(2)$ color space, C_μ is an abelian gauge potential in the n^a direction in the color space and W_μ^a is orthogonal to n^a in the color space, $n^a W_\mu^a = 0$. Further, $n_\mu^a \equiv \partial_\mu n^a$ (spacetime indices on scalar fields will always denote partial derivatives). In the following, we shall refer to the three terms at the rhs of equation (2) as the C-term, the n-term and the W-term, respectively.

In order to guarantee the correct gauge transformation properties

$$\delta n^a = \epsilon^{abc} n^b \alpha^c \qquad \delta W_\mu^a = \epsilon^{abc} W_\mu^b \alpha^c \qquad \delta C_\mu = n^a \alpha_\mu^a, \tag{3}$$

under the gauge transformation,

$$\delta A_\mu^a = (D_\mu \alpha)^a \equiv \alpha_\mu^a + \epsilon^{abc} A_\mu^b \alpha^c, \tag{4}$$

the following constraint has to be imposed,

$$\partial^\mu W_\mu^a + C_\mu \epsilon^{abc} n^b W_\mu^c + n^a W_\mu^b n_\mu^b \equiv 0. \tag{5}$$

In addition, this constraint makes that the number of degrees of freedom of the gauge field and of the decomposition matches.

We now want to restrict to a specific class of gauge fields where we set the W-term (the valence field) equal to zero

$$W_\mu^a = 0, \tag{6}$$

i.e., we assume the restriction

$$\hat{A}_\mu^a = n^a C_\mu + \epsilon^{abc} n_\mu^b n^c. \tag{7}$$

This restriction is gauge invariant, because W_μ^a transforms homogeneously under gauge transformations. Further, the restricted potential $\hat{A}_\mu^a$ still transforms like a $SU(2)$ gauge potential under gauge transformations. The corresponding field strength is, nevertheless, abelian; therefore the above restriction also provides a gauge invariant definition of the abelian projection [11]. Note that this abelian projection is also compatible with the constraint (5).

In a first step, we further restrict to gauge potentials which are solely described by the unit vector n^a, i.e., we set

$$C_\mu = 0, \qquad W_\mu^a = 0. \tag{8}$$

This choice is no longer gauge invariant and also further reduces the number of degrees of freedom. Later on we shall allow for the more general gauge potentials (7) with $C_\mu \neq 0$, and we will also discuss in more detail the issue of gauge transformations; see section 5. It is sometimes assumed that n^a describes the low-energy degrees of freedom of the Yang–Mills field (which was in fact one motivation for the decomposition), but we shall not be concerned with this here.

Inserting the decomposition (2) with the restriction (8) into the Yang–Mills dilaton Lagrangian (1) we arrive at the Lagrangian

$$\mathcal{L} = \tfrac{1}{4} \left(2\partial^\mu \xi \partial_\mu \xi - e^{-2\kappa\xi} H^{a\mu\nu} H_{\mu\nu}^a \right) \tag{9}$$

where

$$H_{\mu\nu}^a \equiv \epsilon^{abc} n_\mu^b n_\nu^c = n^a H_{\mu\nu}, \qquad H_{\mu\nu} \equiv \epsilon^{abc} n^a n_\mu^b n_\nu^c. \tag{10}$$

Remark. We shall use this Lagrangian (9) in section 3 to derive the corresponding Euler–Lagrange equations. Here, of course, the question arises whether these equations are really equivalent to the original Euler–Lagrange equations of the Yang–Mills dilaton theory after the decomposition is inserted into these original equations. We prove in the appendix that this is indeed the case.

For later convenience we prefer to replace the three-component unit vector field n^a by the complex scalar field u via stereographic projection,

$$n^a = \frac{1}{1+|u|^2}(u + \bar{u}, -i(u - \bar{u}), 1 - u\bar{u}); \qquad u = \frac{n_1 + in_2}{1 + n_3}. \tag{11}$$

Then we get for the above Lagrangian (9)

$$H_{\mu\nu} = 2i \frac{u_\mu \bar{u}_\nu - u_\nu \bar{u}_\mu}{(1 + u\bar{u})^2} \tag{12}$$

and

$$\mathcal{L} = \frac{1}{2} \partial^\mu \xi \partial_\mu \xi - 2 e^{-2\kappa\xi} \frac{(u^\mu \bar{u}_\mu)^2 - (u_\mu)^2 (\bar{u}_\nu)^2}{(1 + u\bar{u})^4}. \tag{13}$$

This Lagrangian is integrable in the sense that it has infinitely many symmetries and infinitely many conserved currents. Indeed, it is of the type c of table 3 of [12] and, therefore, has the infinitely many conserved currents

$$J_\mu^{\tilde{G}} = \mathrm{i}\tilde{G}_a(u\pi_\mu - \bar{u}\bar{\pi}_\mu) \tag{14}$$

where

$$\pi_\mu \equiv \frac{\partial \mathcal{L}}{\partial u^\mu}, \qquad \bar{\pi}_\mu \equiv \frac{\partial \mathcal{L}}{\partial \bar{u}^\mu} \tag{15}$$

are the usual canonical four-momenta and $\tilde{G} = \tilde{G}(a)$ ($a \equiv u\bar{u}$) is an arbitrary real function of its argument.

Remark. The Lagrangian (13) has, in fact, an even larger symmetry. It belongs to a rather special class which has not been classified explicitly in [12]. Indeed, this Lagrangian belongs both to type c and to type b of table 3 of [12], i.e., it depends on the target space variable $a \equiv u\bar{u}$ solely via the target space metric function g,

$$\mathcal{L} = \mathcal{F}(g^2 c, d) \tag{16}$$

where

$$g \equiv \mathrm{e}^{-\kappa\xi}(1 + a)^{-2} \qquad a \equiv u\bar{u} \tag{17}$$

$$c \equiv (u^\mu \bar{u}_\mu)^2 - (u_\mu)^2(\bar{u}_\nu)^2, \qquad d \equiv \xi^\mu \xi_\mu. \tag{18}$$

Further, the metric function is of the product form

$$g = g^{(1)}(a)g^{(2)}(\xi). \tag{19}$$

As a consequence, the Lagrangian (13) has the infinitely many conserved currents

$$J_\mu^{\mathcal{G}} = \frac{\mathrm{i}}{g^{(1)}(a)}[\mathcal{G}_{\bar{u}}\pi_\mu - \mathcal{G}_u\bar{\pi}_\mu] \tag{20}$$

where $\mathcal{G} = \mathcal{G}(u, \bar{u})$ is an arbitrary real function of its arguments. From a geometric point of view the existence of these conserved currents is quite obvious, because they are just the Noether currents of the area-preserving diffeomorphisms on the target two-sphere which is spanned by the complex field u, and these are symmetries of the above Lagrangian; see, e.g., [13–16].

3. Static solutions

The energy functional for static configurations which corresponds to the Lagrangian (13) reads

$$E = \int \mathrm{d}^3\mathbf{r} \left(\frac{1}{2}\nabla\xi \cdot \nabla\xi + 2\,\mathrm{e}^{-2\kappa\xi} \frac{(\nabla u \cdot \nabla\bar{u})^2 - (\nabla u)^2(\nabla\bar{u})^2}{(1 + u\bar{u})^4} \right). \tag{21}$$

Momentarily, we want to switch to the more general class of energy functionals

$$E = \int \mathrm{d}^3\mathbf{r} \left(\frac{1}{2}\nabla\xi \cdot \nabla\xi + 2G^{(1)}(a)G^{(2)}(\xi)[(\nabla u \cdot \nabla\bar{u})^2 - (\nabla u)^2(\nabla\bar{u})^2] \right) \tag{22}$$

because this general class has exactly the same integrability properties and may be solved in exactly the same way. Later we will specialize to the case

$$\begin{aligned} g^{(1)}(a) &= (1 + a)^{-2}, & G^{(1)} &\equiv (g^{(1)})^2 \\ g^{(2)}(\xi) &= \mathrm{e}^{-\kappa\xi}, & G^{(2)} &\equiv (g^{(2)})^2 \end{aligned} \tag{23}$$

when it is needed.

First of all, let us observe that the energy functional (22) is the sum of a term which is quadratic in first derivatives and another term which is quartic. Therefore, the Derrick criterion does not rule out the existence of static finite energy solutions, and we will indeed find that they exist. The Euler–Lagrange equation for the variation w.r.t. $\bar{u}$ is

$$\nabla \cdot (g^{(1)}(a)G^{(2)}(\xi)\vec{K}) = 0 \tag{24}$$

where

$$\vec{K} \equiv (\nabla u \cdot \nabla \bar{u})\nabla u - (\nabla u)^2 \nabla \bar{u} \tag{25}$$

(observe the appearance of $g^{(1)} \equiv \sqrt{G^{(1)}}$ in the equation). The Euler–Lagrange equation for the variation w.r.t. ξ is

$$\Delta \xi = 2G^{(1)}G_\xi^{(2)}[(\nabla u \cdot \nabla \bar{u})^2 - (\nabla u)^2(\nabla \bar{u})^2]. \tag{26}$$

Next, we introduce spherical polar coordinates $\vec{r} = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta)$ and the corresponding frame of unit basis vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi)$, and we employ the ansatz

$$\xi = \xi(r), \qquad u = v(\theta)\,e^{im\varphi}. \tag{27}$$

Then the Euler–Lagrange equation (24) simplifies to

$$\nabla \cdot (g^{(1)}(a)\vec{K}) = 0 \tag{28}$$

because $\nabla\xi \cdot \vec{K} = 0$. With

$$\nabla u = \frac{1}{r}\,e^{im\varphi}\left(v_\theta \hat{e}_\theta + \frac{imv}{\sin\theta}\hat{e}_\varphi\right) \tag{29}$$

and

$$\vec{K} = \frac{2}{r^3}\,e^{im\varphi}\left(\frac{m^2v^2v_\theta}{\sin^2\theta}\hat{e}_\theta + \frac{imvv_\theta^2}{\sin\theta}\hat{e}_\varphi\right) \tag{30}$$

and using that $g^{(1)}(a(\theta))$ is a function of θ only, we find

$$\nabla \cdot (g^{(1)}(a)\vec{K}) = \frac{2m^2\,e^{im\varphi}}{r^4\sin\theta}\partial_\theta\left(\frac{g^{(1)}vv_\theta}{\sin\theta}\right) \equiv 0 \tag{31}$$

and, therefore, the first trivial integral

$$\frac{g^{(1)}vv_\theta}{\sin\theta} = \mu = \text{const..} \tag{32}$$

The further evaluation depends on the explicit form of $g^{(1)}$. Choosing $g^{(1)} = (1+a)^{-2} = (1+v^2)^{-2}$ we get

$$\frac{V_\theta}{(1+V)^2} = \mu \sin\theta \qquad V \equiv v^2 \tag{33}$$

and the solution

$$V = \frac{1 - \mu\cos\theta + \lambda}{\mu\cos\theta - \lambda} \tag{34}$$

where λ and μ are the two integration constants. The integration constants are fixed by the requirement that u should be a genuine map $S^2 \to S^2$. This requires that the modulus v of u should cover the whole positive real semiaxis, i.e.,

$$V(0) = 0, \qquad V(\pi) = \infty. \tag{35}$$

This leads to

$$\lambda = -\tfrac{1}{2}, \qquad \mu = \tfrac{1}{4} \tag{36}$$

and to

$$V = \frac{1 - \cos\theta}{1 + \cos\theta}, \qquad u = \tan\frac{\theta}{2}\, e^{im\varphi}. \tag{37}$$

Indeed, the corresponding u describes a map $S^2 \to S^2$ with the winding number m. The resulting field strength H_{jk} just describes the abelian magnetic monopole with charge m. Indeed, for the hodge dual vector

$$H^i = \epsilon^{ijk} H_{kj} \tag{38}$$

we get

$$\vec{H} = \frac{m}{r^2}\hat{e}_r. \tag{39}$$

The magnetic charge $m \in \mathbb{Z}$ is quantized by the topological nature of u.

In order to solve the Euler–Lagrange equation (26), we first need the expression

$$(\nabla u \cdot \nabla\bar{u})^2 - (\nabla u)^2(\nabla\bar{u})^2 = \frac{4m^2 v^2 v_\theta^2}{r^4 \sin^2\theta} \tag{40}$$

for the ansatz (27). We find for the Euler–Lagrange equation (26)

$$\frac{1}{r^2}\partial_r(r^2\partial_r\xi) = \frac{8m^2 v^2 v_\theta^2}{r^4 \sin^2\theta}(g^{(1)})^2 G_\xi^{(2)} = \frac{8m^2\mu^2}{r^4}G_\xi^{(2)} \tag{41}$$

where we used the first integral (32). With the new variable $s = r^{-1}$ we get

$$\xi_{ss} = 8m^2\mu^2 G_\xi^{(2)} \tag{42}$$

and, upon multiplication with ξ_s, the first integral

$$\xi_s^2 = 16m^2\mu^2 G^{(2)} + \tilde{\lambda} \tag{43}$$

where $\tilde{\lambda}$ is an integration constant. This expression may easily be integrated, after taking the square root, and results in

$$s + s_0 = \int \frac{d\xi}{\sqrt{16m^2\mu^2 G^{(2)} + \tilde{\lambda}}}, \tag{44}$$

and s_0 is another integration constant. For a further evaluation, one has to choose an explicit function for $G^{(2)}$. Choosing $G^{(2)} = e^{-2\kappa\xi}$, and $\mu = (1/4)$, we get

$$s + s_0 = \int \frac{d\xi}{\sqrt{m^2 e^{-2\kappa\xi} + \tilde{\lambda}}}. \tag{45}$$

For finite energy solutions one has to choose $\tilde{\lambda} = 0$, in which case the integral is trivial and has the solution

$$s + s_0 = \frac{e^{\kappa\xi}}{\kappa|m|} \quad \Rightarrow \quad \xi = \frac{1}{\kappa}\ln[|m|\kappa(s + s_0)]. \tag{46}$$

For a further evaluation, we have to impose boundary conditions. We require that ξ covers the whole positive real semi-axis which implies that

$$\xi(s = 0) = 0, \qquad \xi(s = \infty) = \infty \qquad \Rightarrow \qquad s_0 = \frac{1}{\kappa|m|} \tag{47}$$

and, therefore

$$\xi = \frac{1}{\kappa}\ln[|m|\kappa s + 1]. \tag{48}$$

These solutions are precisely the limiting solutions (in the limit where the number of nodes of a certain ansatz function in the numerical analysis goes to infinity) which have been found in [2] (for $m = 1$) and in [3] (for general m).

For the energy we find (remember $s \equiv r^{-1}$)

$$
\begin{aligned}
E &= \int \mathrm{d}^3\mathbf{r} \left(\frac{1}{2} \nabla \xi \cdot \nabla \xi + 2\,\mathrm{e}^{-2\kappa\xi} \frac{(\nabla u \cdot \nabla \bar{u})^2 - (\nabla u)^2 (\nabla \bar{u})^2}{(1 + u\bar{u})^4} \right) \\
&= \frac{1}{2} \int \mathrm{d}^3\mathbf{r} \left(\xi_r^2 + \frac{m^2}{r^4}\,\mathrm{e}^{-2\kappa\xi} \right) \\
&= \frac{1}{2} 4\pi \int \mathrm{d}r\, r^2 \left(\frac{m^2}{(|m|\kappa r + r^2)^2} + \frac{m^2}{r^4} \frac{1}{(|m|\kappa r^{-1} + 1)^2} \right) \\
&= 4\pi m^2 \int \frac{\mathrm{d}r}{(|m|\kappa + r)^2} = \frac{4\pi |m|}{\kappa}.
\end{aligned}
\tag{49}
$$

Therefore, the energy is linear in the topological charge, as was already observed in [3]. This gives rise to the question whether there exists a Bogomolny-type bound in the integrable submodel. And that is indeed the case, as we shall see in the following section.

4. The Bogomolny bound

We introduce the vector

$$
\vec{H} = 2\mathrm{i} \frac{(\nabla u) \times (\nabla \bar{u})}{(1 + u\bar{u})^2} = \frac{1}{2} \epsilon^{abc} n^a (\nabla n^b) \times (\nabla n^c)
\tag{50}
$$

and express the energy functional like

$$
\begin{aligned}
E &= \frac{1}{2} \int \mathrm{d}^3\mathbf{r} [(\nabla \xi)^2 + G^{(2)}(\xi)\vec{H}^2] \\
&= \frac{1}{2} \int \mathrm{d}^3\mathbf{r} (\nabla \xi - g^{(2)}\vec{H})^2 + \int \mathrm{d}^3\mathbf{r}\, g^{(2)} \nabla \xi \cdot \vec{H} \\
&\geqslant \int \mathrm{d}^3\mathbf{r}\, g^{(2)} \nabla \xi \cdot \vec{H} \equiv E_{\mathrm{Bog}}.
\end{aligned}
\tag{51}
$$

Therefore, the Bogomolny equation is

$$
\nabla \xi - g^{(2)}\vec{H} = 0.
\tag{52}
$$

We now show that our solutions (37), (48) for the ansatz (27) obey this Bogomolny equation. For the ansatz we have $\nabla \xi = \xi_r \hat{\mathbf{e}}_r$ and

$$
\vec{H} = \frac{4m}{r^2 \sin\theta} \frac{v v_\theta}{(1 + v^2)^2} \hat{\mathbf{e}}_r = \frac{4m}{r^2} \frac{g^{(1)} v v_\theta}{\sin\theta} \hat{\mathbf{e}}_r = \frac{m}{r^2} \hat{\mathbf{e}}_r
\tag{53}
$$

where we used (32) and $\mu^{-1} = 4$; see (36). The Bogomolny equation now becomes

$$
\xi_r = \frac{m}{r^2} g^{(2)}
\tag{54}
$$

or, after the variable change $s = r^{-1}$ and squaring of the resulting expression,

$$
\xi_s^2 = m^2 G^{(2)}
\tag{55}
$$

which is exactly the first integral (43) for the special choice $\tilde{\lambda} = 0$ of the integration constant $\tilde{\lambda}$. This is, of course, consistent with our remark that finite energy solutions require $\tilde{\lambda} = 0$. In short, the static solutions of the last section indeed obey the Bogomolny equation (52).

It remains to evaluate the Bogomolny energy in the Bogomolny inequality (51). We get

$$E_{\text{Bog}} = \int \mathrm{d}^3\mathbf{r}\, g^{(2)} \nabla \xi \cdot \vec{H} = \int \mathrm{d}^3\mathbf{r}\, \nabla \tilde{g} \cdot \vec{H} \tag{56}$$

where $\tilde{g}$ obeys

$$\tilde{g}_\xi = g^{(2)}(\xi) \tag{57}$$

Specifically, for our model $g^{(2)} = \mathrm{e}^{-\kappa\xi}$ and, therefore,

$$E_{\text{Bog}} = -\frac{1}{\kappa} \int \mathrm{d}^3\mathbf{r}\, (\nabla \mathrm{e}^{-\kappa\xi}) \cdot \vec{H}. \tag{58}$$

This we want to compare now with an expression for the winding number Q of a map $\mathbb{R}_0^3 \to S^3$, where $\mathbb{R}_0^3$ is one-point compactified Euclidean space. A useful expression for our purpose is (see, e.g., [17], p 24, equation (1.37))

$$Q = -\frac{1}{4\pi^2} \int \mathrm{d}^3\mathbf{r}\, \sin^2 \Theta (\nabla\Theta) \cdot \epsilon^{abc} n^a (\nabla n^b) \times (\nabla n^c)$$

$$= -\frac{1}{2\pi^2} \int \mathrm{d}^3\mathbf{r}\, \sin^2 \Theta (\nabla\Theta) \cdot \vec{H}. \tag{59}$$

Here, if the unit target space three-sphere is spanned by a unit four-vector $X^\alpha, \alpha = 1, \ldots, 4$, then the meaning of the target space coordinates n^a, Θ is

$$X^a = n^a \sin\Theta, \qquad a = 1, 2, 3, \qquad X^4 = \cos\Theta, \tag{60}$$

and the compactification condition may be chosen, e.g.,

$$\lim_{\mathbf{r}\to\infty} \Theta = \pi, \tag{61}$$

which means that infinity is mapped to the south pole of the three-sphere. Comparing the two expressions we identify

$$g^{(2)} \equiv \mathrm{e}^{-\kappa\xi} = \frac{1}{\pi}\left(\Theta - \frac{1}{2}\sin 2\Theta\right) \quad \Rightarrow \quad \nabla g^{(2)} = \frac{2}{\pi}\sin^2\Theta\nabla\Theta \tag{62}$$

and the compactification condition becomes

$$\lim_{\mathbf{r}\to\infty} g^{(2)} = 1 \tag{63}$$

which is precisely the boundary condition $\xi(r = \infty) = \infty$; see equation (47).

Therefore, we find

$$E_{\text{Bog}} = \frac{4\pi}{\kappa} Q \tag{64}$$

and the Bogomolny energy may indeed be expressed by a topological charge. In the case of our static solutions of section 3 it is the field u which describes a map $S^2 \to S^2$ with winding number $|m|$ and, therefore, provides the nontrivial winding number $Q = |m|$. With this identification, expression (49) for the energies of the static configurations is exactly equal to the Bogomolny energy, which we wanted to prove.

5. The more general system

We now want to study the case of the more general gauge field (7) with $C_\mu \neq 0$. The Yang–Mills dilaton Lagrangian then reduces to

$$\mathcal{L} = \frac{1}{4}\left(2\partial^\mu\xi\partial_\mu\xi - \mathrm{e}^{-2\kappa\xi}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}\right), \tag{65}$$

$$\hat{F}_{\mu\nu} \equiv F_{\mu\nu} - H_{\mu\nu}, \tag{66}$$

where

$$F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu \tag{67}$$

is the field strength of the abelian gauge field C_μ and $H_{\mu\nu}$ is defined in (12).

5.1. General remarks

It may appear from expression (65) that the system with $C_\mu \neq 0$ is equivalent to the system with $C_\mu = 0$ and may be described by simply replacing $H_{\mu\nu}$ by $\hat{F}_{\mu\nu}$, but this is not true. The important point here is that $H_{\mu\nu}$ as an antisymmetric 4×4 matrix is of second rank. Therefore, if $H_{\mu\nu}$ is interpreted as an electromagnetic field tensor, it corresponds to fields such that the electric and magnetic fields are perpendicular, $\vec{E} \cdot \vec{B} = 0$ (the so-called 'radiation fields'). On the other hand, no condition is imposed on the abelian gauge field C_μ; therefore $F_{\mu\nu}$ generically is a non-degenerate fourth rank matrix. Observe that one condition $\vec{E} \cdot \vec{B} = 0$ is sufficient to reduce from a fourth rank matrix to a second rank one. The reason is that the eigenvalues of an antisymmetric matrix always come in pairs $\pm i\lambda$, so one rank-reducing condition always sets two eigenvalues equal to zero and reduces the rank by two. As a consequence, any radiation field may be locally described by a tensor $H_{\mu\nu}$. Globally, the set of radiation fields that can be described by a tensor $H_{\mu\nu}$ is more restricted, at least as long as one requires that n^a is a globally well-defined map from $\mathbb{R}^3$ to S^2 (see e.g. [18]).

It is, however, possible to remove the *n*-term in the expression of the gauge potential by a gauge transformation. Indeed, choose the unit vector field m^a, then the gauge transformation

$$U = \sqrt{\frac{1 + \cos\gamma}{2}} - \frac{i}{\sqrt{2(1 + \cos\gamma)}} \alpha^a \sigma^a \tag{68}$$

$$\alpha^a \equiv \epsilon^{abc} n^b m^c, \qquad \cos\gamma = n^a m^a \tag{69}$$

transforms the unit vector field n^a into the new unit vector field m^a (more precisely, the matrix $\mathbf{n} \equiv n^a \sigma^a$ into the matrix $\mathbf{m} \equiv m^a \sigma^a$). If we now choose $m^a = $ const., then the *n*-term is absent in the gauge-transformed gauge potential, because $m^a_\mu = 0$. Specifically, for $m^a = \hat{e}^a_3 \equiv \delta^{a3} = (0, 0, 1)$, the corresponding gauge transformation (68) is

$$U = \sqrt{\frac{1 + n^3}{2}} - \frac{i}{\sqrt{2(1 + n^3)}} (n^2 \sigma^1 - n^1 \sigma^2). \tag{70}$$

This expression is not well defined at $n^a = (0, 0, -1)$ (its value depends on the order in which the limit $n^1 \to 0, n^2 \to 0$ is performed). A topologically nontrivial n^a covers the whole target space S^2 and, therefore, also the value $(0, 0, -1)$. The corresponding U is, therefore, singular. If the field strength $H_{\mu\nu}$ corresponding to n^a is regular, however, there always exists a further gauge transformation $V = \exp(i\beta\sigma^3)$ (for some appropriate β not expressible in terms of n^a only) which leaves $\hat{e}^a_3 = (0, 0, 1)$ invariant, such that the composition VU is regular everywhere. (Observe that this does not happen for the solutions of section 3, because there the field strength is the singular magnetic monopole field.)

Next, we want to calculate how a gauge potential transforms under this gauge transformation. If we define a general Lie-algebra valued gauge potential as

$$A_\mu = \tfrac{1}{2} A^a_\mu \sigma^a \tag{71}$$

then this gauge potential transforms like

$$A_\mu \to U A_\mu U^\dagger - iU \partial_\mu U^\dagger \tag{72}$$

(the factor $-i$ in front of the inhomogeneous term is due to the fact that we use the Hermitian basis $(1/2)\sigma^a$ in the Lie algebra instead of the anti-Hermitian $t^a = -(i/2)\sigma^a$). Specifically, if we choose for the untransformed gauge potential

$$A^a_\mu = \epsilon^{abc} n^b_\mu n^c \tag{73}$$

consisting only of the *n*-term, and the transformation (70) for the gauge transformation, then the transformed gauge potential is

$$A'^a_\mu = \frac{n^2_\mu n^1 - n^1_\mu n^2}{1 + n^3} \hat{e}^a_3 \tag{74}$$

that is, only the C_μ-term is nonzero, and the transformed unit vector is $m^a = \hat{e}^a_3$. If this gauge potential is singular (because n^a is topologically nontrivial) but leads to a non-singular field strength, then there always exists a further gauge transformation (the above-mentioned V) which transforms A'^a_μ to a regular gauge potential. This transformation acts as an abelian gauge transformation on the abelian gauge field C_μ, i.e.

$$V : \quad \frac{n^2_\mu n^1 - n^1_\mu n^2}{1 + n^3} \quad \rightarrow \quad \frac{n^2_\mu n^1 - n^1_\mu n^2}{1 + n^3} + \beta_\mu. \tag{75}$$

Now we want to discuss the conservation laws of the Lagrangian (65). For this purpose it is useful to rewrite it like (remember $c \equiv (u^\mu \bar{u}_\mu)^2 - (u_\mu)^2 (\bar{u}_\nu)^2$)

$$\mathcal{L} = \frac{1}{2}\partial^\mu \xi \partial_\mu \xi - e^{-2\kappa\xi} \left(\frac{2c}{(1 + u\bar{u})^4} - i\frac{u_\mu \bar{u}_\nu - u_\nu \bar{u}_\mu}{(1 + u\bar{u})^2} F^{\mu\nu} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right). \tag{76}$$

This Lagrangian still has the infinitely many conserved currents (20), because the term $(1+u\bar{u})^{-2}(u_\mu \bar{u}_\nu - u_\nu \bar{u}_\mu)$ is still invariant under the corresponding target space transformations $\delta u = i(1 + u\bar{u})^2 \mathcal{G}_{\bar{u}}$, etc, as may be checked easily. But the above Lagrangian has even more conserved currents, as we demonstrate now. For this purpose, we rewrite it in the slightly more general form

$$\mathcal{L} = \frac{1}{2}\partial^\mu \xi \partial_\mu \xi - G^{(2)}(\xi) \left(2G^{(1)}(a)c - ig^{(1)}(a)(u_\mu \bar{u}_\nu - u_\nu \bar{u}_\mu) F^{\mu\nu} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \tag{77}$$

like in (22), (remember $G^{(1)} \equiv (g^{(1)})^2$), and calculate the following Euler–Lagrange equations, for $\bar{u}$

$$2\partial_\mu (G^{(2)} g^{(1)} K^\mu) + ig^{(1)} \partial_\mu (G^{(2)} F^{\mu\nu} u_\nu) = 0, \tag{78}$$

for ξ,

$$\xi^\mu_\mu + G^{(2)}_\xi \left(2G^{(1)}(a)c - ig^{(1)}(a)(u_\mu \bar{u}_\nu - u_\nu \bar{u}_\mu) F^{\mu\nu} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) = 0, \tag{79}$$

for C_μ,

$$\partial^\mu [G^{(2)}(F_{\mu\nu} - 2ig^{(1)}(u_\mu \bar{u}_\nu - u_\nu \bar{u}_\mu))] = 0. \tag{80}$$

It follows easily from the last field equation that the currents

$$j^{\mathcal{H}}_\mu = \mathcal{H}(\xi) G^{(2)}(\xi)(F_{\mu\nu} - 2ig^{(1)}(u_\mu \bar{u}_\nu - u_\nu \bar{u}_\mu))\xi^\nu \tag{81}$$

are conserved, where $\mathcal{H}(\xi)$ is an arbitrary real function of its argument. These currents are completely analogous to the additional conserved currents which were found for abelian gauge theories in [19]; see equations (61)–(63) of that paper.

Remark. Setting $u = 0$ and observing that $F_{\mu\nu}$ is a general abelian gauge field strength, it follows that the general Maxwell dilaton system has infinitely many conserved currents.

5.2. Further solutions

A first possibility to construct further solutions is provided by just adding a pure gauge *C*-term, i.e. a term $(\partial_\mu \lambda) n^a$ to a given solution. This is a trivial modification from the point of view of the integrable submodel, because its Lagrangian only depends on the field strength $F_{\mu\nu}$ (see (65), but it is a nontrivial modification from the point of view of the full gauge potential

(with non-zero W-term), because the Lagrangian of the full theory depends on the abelian gauge potential C_μ (and not only on the abelian field strength). The term $(\partial_\mu \lambda)n^a$ can be removed by a nonabelian gauge transformation (with gauge parameter $\alpha^a = \lambda n^a$) for a general gauge potential, but this transformation acts nontrivially on the W-term and so just transfers the physical effect of $(\partial_\mu \lambda)n^a$ from the C-term to the W-term.

Specifically, by choosing $\lambda = q x^4$ (here q is a constant, x^4 is the Euclidean 'time', and we momentarily switch to Euclidean spacetime conventions to be consistent with [20]), i.e., by adding a C-term with $C_\mu = q\delta_{\mu4}$ to the monopole-type solutions of section 3, we are able to reproduce the special analytic solution of [20] (they are called 'dyonic-type generalizations of the monopole solutions' in that paper; see their equation (17); for the constant q they use the symbol $\bar{u}$, which we do not employ here for obvious reasons). Obviously, these solutions have the same energies and fulfill the same Bogomolny bounds as the solutions of section 3.

We may try to find further solutions by solving the full system of field equations (78)–(80). We assume that no field variable depends on time (static solutions), and we further assume that C_μ only describes an electric field (i.e., $F_{0j} = E_j$, $F_{jk} = 0$; observe that $H_{\mu\nu} \rightarrow H_{jk}$ is automatically purely magnetic for static u). Under these assumptions, the second term in the field equation for $\bar{u}$, equation (78), does not contribute, and we get equation (24) of section 3. Also the term $(u_\mu \bar{u}_\nu - u_\nu \bar{u}_\mu)F^{\mu\nu}$ in equation (79) is zero under these assumptions. Now we further restrict to

$$u = v(\theta)\, e^{im\varphi}, \qquad \xi = \xi(r), \qquad \vec{E} = E(r)\hat{e}_r \tag{82}$$

and find for u exactly the same solutions as in section 3 (i.e., the first integral (32) for general $g^{(1)}$, and the generalized hedge-hogs (37) for $g^{(1)} = (1 + u\bar{u})^{-2}$). For these solutions for u, the second term in equation (80) is zero, and equation (80) simplifies to

$$\nabla(G^{(2)}E(r)\hat{e}_r) = 0 \tag{83}$$

with the solution

$$E(r) = \frac{q}{4\pi r^2 G^{(2)}} \tag{84}$$

where q is the constant electrical charge. Finally, equation (79) leads to the first integral

$$\xi_s^2 = 16m^2\mu^2 G^{(2)} + \frac{q^2}{4\pi^2 G^{(2)}} + \tilde{\lambda} \tag{85}$$

where $s \equiv (1/r)$ and the calculation is completely analogous to the calculation in section 3. For a further evaluation we have to specify $G^{(2)}(\xi)$. For the dilaton case $G^{(2)} = \exp(-2\kappa\xi)$ and the generalized hedge-hogs for u we get (remember $\mu^{-1} = 4$)

$$\xi_s^2 = m^2\, e^{-2\kappa\xi} + \frac{q^2}{4\pi^2}\, e^{2\kappa\xi} + \tilde{\lambda}. \tag{86}$$

Unfortunately, there does not exist a physically acceptable (finite energy) solution if both m and q are different from zero. Choosing $q = 0$ we recover the solutions of section 3, but choosing $m = 0$, we find different, purely electric solutions. Finite energy requires again $\tilde{\lambda} = 0$, and we get

$$\xi_s^2 = \frac{q^2}{4\pi^2}\, e^{2\kappa\xi}. \tag{87}$$

Comparing with the equation

$$\xi_s^2 = m^2\, e^{-2\kappa\xi} \tag{88}$$

for the magnetic solutions of section 3, we see that we recover the electric equation by the replacements $m \to (q/2\pi)$ and $\xi \to -\xi$, therefore the electric solution for ξ is

$$\xi = -\frac{1}{\kappa} \ln \left(\frac{|q|}{2\pi} \kappa s + 1 \right). \tag{89}$$

Also the energy of the electric solution (84), (89) is the same as the energy of the magnetic solution (49) after the replacement $m \to (q/2\pi)$. The electric solutions are also Bogomolny solutions, and the Bogomolny bound may be derived exactly like in section 4, where one only has to replace the 'magnetic' vector $\vec{H}$ by the electric vector $\vec{E} = -\nabla C_0$.

Remark. A Bogomolny bound does not exist when both electric and magnetic vectors are present.

Remark. The electric solution (84) and (89) has been derived in [2] from the simplest magnetic solution in a slightly different way, by employing the nontrivial duality symmetry

$$F^a_{\mu\nu} \to e^{-2\kappa\xi} \tilde{F}^a_{\mu\nu}, \qquad \xi \to -\xi \tag{90}$$

which is present for the Yang–Mills dilaton theory (here $\tilde{F}^a_{\mu\nu}$ is the Hodge dual of the nonabelian field strength $F^a_{\mu\nu}$).

5.3. Further generalizations

Finally, let us briefly discuss some generalizations of the integrable Lagrangians discussed so far, which are still integrable (have infinitely many symmetries) and allow for the ansatz (27) in spherical polar coordinates, and for trivial first integrals of the resulting ODEs. One class of models is given by the Lagrangians (with non-polynomial kinetic terms)

$$\mathcal{L} = \tfrac{1}{2}(\xi_\mu \xi^\mu)^\alpha - 2G^{(2)}(\xi)G^{(1)}(a)((u^\mu \bar{u}_\mu)^2 - (u_\mu)^2(\bar{u}_\nu)^2)^\beta \tag{91}$$

where α and β are parameters. This model has the infinitely many conserved currents (analogously to the currents (20))

$$J^{\mathcal{G}}_\mu = i(G^{(1)})^{-\frac{\beta}{2}}[\mathcal{G}_{\bar{u}}\pi_\mu - \mathcal{G}_u \bar{\pi}_\mu] \tag{92}$$

where $\mathcal{G} = \mathcal{G}(u, \bar{u})$ is an arbitrary real function of its arguments. Further, the separation of variables ansatz (27) is compatible with the Euler–Lagrange equations, and the resulting ODEs are solvable by quadratures. Whether there exist finite energy static solutions depends both on the choice of the parameters α and β (where Derrick's theorem provides a selection criterion) and on the functions $G^{(1)}$ and $G^{(2)}$. In the special case $\alpha = \frac{3}{2}, \beta = \frac{3}{4}$, the energy functional for static solutions enjoys an enhanced base space symmetry in $\mathbb{R}^3$ (conformal symmetry instead of just Galilean symmetry) and, therefore, a separation of variables ansatz in toroidal coordinates is compatible with the field equations and leads to static solutions, quite analogously to the case of the model of Aratyn, Ferreira and Zimerman [14, 21, 22].

Another generalization is given by

$$\begin{aligned}
\mathcal{L} = \tfrac{1}{2}\partial^\mu\xi\partial_\mu\xi - 2G^{(2)}(\xi)2G^{(1)}(a)c \\
- iH^{(2)}(\xi)H^{(1)}(a)(u_\mu\bar{u}_\nu - u_\nu\bar{u}_\mu)F^{\mu\nu} + \tfrac{1}{4}K^{(2)}(\xi)F^{\mu\nu}F_{\mu\nu}.
\end{aligned} \tag{93}$$

If $G^{(1)} = (H^{(1)})^2$, then this Lagrangian has the infinitely many conserved currents (20)); if this condition does not hold, only the smaller set of currents (14) is conserved. Further, the currents

$$j^{\mathcal{H}}_\mu = \mathcal{H}(\xi)(K^{(2)}(\xi)F_{\mu\nu} - 2iH^{(1)}H^{(2)}(u_\mu\bar{u}_\nu - u_\nu\bar{u}_\mu))\xi^\nu \tag{94}$$

are conserved, where $\mathcal{H}(\xi)$ is an arbitrary real function of its argument. The separation of variable ansatz (82) for static solutions and a purely electric $F_{\mu\nu}$ is again compatible with the Euler–Lagrange equations, and the resulting ODEs are again solvable by quadratures. The existence of finite energy static solutions depends again on the choice of the arbitrary functions. For instance, there exist infinitely many solutions with both non-zero magnetic and electric fields for the following choice,

$$G^{(1)} = (1 + a)^{-4}, \qquad G^{(2)} = (K^{(2)})^{-1} = e^{-2\kappa\xi}. \tag{95}$$

Here, the only difference to the Yang–Mills dilaton case is that $(K^{(2)})^{-1} = e^{-2\kappa\xi}$ instead of $K^{(2)} = e^{-2\kappa\xi}$ (the values for $H^{(1)}$, $H^{(2)}$ are irrelevant because the term multiplying them does not contribute to purely magnetic $H_{\mu\nu}$ and purely electric $F_{\mu\nu}$).

6. Discussion

It has been one of the initial motivations of this investigation to shed more light on previous results on the Yang–Mills dilaton theory. In this respect, our first result is the simple observation that Yang–Mills dilaton theory contains an integrable subsystem, i.e., a subsystem with infinitely many target space symmetries and infinitely many conserved currents, and this subsystem is non-empty in the sense that it contains nontrivial (e.g., static finite energy) solutions. This fact is also intimately related to our second result, namely a possible explanation for the existence of infinitely many static analytic solutions. In this context, it is interesting to note the role which is played by the symmetries of the integrable subsystem. The consistency of the ansatz (27) in spherical polar coordinates is explained by the base space symmetries of the model (essentially by rotational symmetry). On the other hand, the solvability of the resulting ODEs by simple quadratures might be related to the integrability of the model, i.e., to the existence of infinitely many target space symmetries and conservation laws. The conjecture that solvability follows from integrability is supported both by the corresponding, rigorous results in lower dimensions and by the fact that it is true in all known examples of higher-dimensional integrable theories. It holds, e.g., for the model of Aratyn, Ferreira and Zimerman [21, 22] where the base space symmetries allow for an ansatz in toroidal coordinates such that the resulting ODEs are solvable by quadratures, or for a class of models similar to that of this paper, but with non-polynomial kinetic energy expressions; see [23]. A more rigorous mathematical analysis of the relation between integrability and solvability for integrable theories in higher dimensions is still missing and would be highly desirable.

At this point a word of caution may be appropriate. Although the relation between infinitely many conservation laws and (at least classical) solvability may carry over from 1+1 to higher dimensions, this is certainly not the case for some other features of 1+1 dimensional integrable theories. In 1+1 dimensional integrable theories one has, for instance, the possibility of conserved charges of arbitrary spin which, in turn, lead to the factorizability of the S-matrix. This cannot happen in higher dimensions due to the Coleman–Mandula theorem [24]. Also the transition from classical to quantum integrable systems will most likely be more complicated in higher dimensions. The algebra of the infinitely many conserved charges or conserved currents, as exposed, e.g. in [12, 15, 16] for certain classes of theories, may, nevertheless, be a good starting point for the quantization. This issue is, however, beyond the scope of the present paper.

Our third result is the explanation of the fact that the energies of our static solutions grow linearly with the topological charge. This is explained by the Bogomolny bound (51) for the integrable submodel and by the observation that the static solutions saturate this bound.

Remark. For the simplest solution with $m = 1$, a Bogomolny-type bound was given in [2]. However, this Bogomolny bound holds for a certain subsector of the spherically symmetric ansatz of the Yang–Mills dilaton system; therefore it only applies to spherically symmetric solutions. On the other hand, our Bogomolny bound (51) holds completely generally for the integrable subsystem and does not require a certain symmetry of the solution.

In section 5 we found similar results for a slightly more general Lagrangian (i.e. for a slightly less restricted gauge potential) still maintaining integrability and solvability. It is interesting that both finite energy solutions and Bogomolny bounds seem to exist only for purely magnetic or purely electric gauge fields, but not for the mixed case. Further, we presented some more general Lagrangians, no longer related to the Yang–Mills dilaton system, which are still both integrable and solvable.

Finally let us point out that, apart from shedding more light on some issues of Yang–Mills dilaton theory, our investigation has an additional interest. Observe that we have used the decomposition of Cho, Faddeev, Niemi and Shabanov—which originally was mainly motivated by the description of the low-energy degrees of freedom of Yang–Mills theory—for a different purpose, namely for the exposure of an intergrable subsector within this theory. Here, it is important to note that the target space transformations—which are symmetry transformations in the submodel—are very simple in terms of the decomposition fields. They are essentially area-preserving diffeomorphisms of the target space two-sphere spanned by the unit vector n^a (or by the complex field u); therefore they are just geometric transformations. On the other hand, in terms of the original Yang–Mills field, they certainly would be rather nontrivial, nonlocal transformations, which would be quite difficult to detect.

This immediately raises the question of generalizations and further applications. First of all, we believe that most likely our investigation carries over without problems to the case of the Einstein–Yang–Mills dilaton system (that is, to an appropriately chosen subsystem thereof), given the striking similarity between the two theories found, e.g. in [2]. Another interesting question is whether the CFNS decomposition or a similar decomposition can be used, e.g., to unravel an integrable subsector to which the sector of self-dual solutions of Yang–Mills theory belongs. These problems are under investigation. In any case, it seems that the use of nonlocal field transformations is a useful instrument for the discovery of nontrivial symmetries and integrable subsectors of nonlinear field theories.

Acknowledgments

Appendix

Here we want to prove that inserting the CFNS decomposition (2) directly into the Yang–Mills dilaton action and deriving the Euler–Lagrange equations with respect to the decomposition fields gives the same field equations as those that are obtained by inserting the decomposition into the Euler–Lagrange equations of the original Yang–Mills dilaton theory. For the case of pure $SU(2)$ Yang–Mills theory this equivalence of the two different ways to derive the field equations was already proven in [7], but as we study a different theory in our paper and use

a slightly different parametrization for the decomposition, we provide the proof here for the convenience of the reader.

The dilaton field is not changed under the decomposition; therefore the equivalence obviously holds for the dilaton field equation. So we focus on the Yang–Mills equation in the following. In terms of the Yang–Mills gauge potential A^a_μ, the corresponding Euler–Lagrange equation reads

$$\mathrm{e}^{-2\kappa\xi}\epsilon^{abc}A^b_\nu F^{c\nu\lambda} + \partial_\nu(\mathrm{e}^{-2\kappa\xi}F^{a\nu\lambda}) = 0. \tag{A.1}$$

In a first step we want to evaluate this expression for the gauge invariant abelian projection, that is, the restriction (7) with both n^a and C_μ nonzero. Inserting this decomposition we easily find the equation

$$n^a\partial_\mu(\mathrm{e}^{-2\kappa\xi}\hat{F}^{\mu\nu}) = 0 \tag{A.2}$$

(where $\hat{F}^{\mu\nu}$ is defined in (66)), and the factor of n^a in front of the expression is obviously irrelevant and may be omitted. This now has to be compared with equations (78) and (80) for the special case $G^{(2)} = \mathrm{e}^{-2\kappa\xi}$, $g^{(1)} = (1 + u\bar{u})^{-2}$. Equation (80) may be written like

$$\partial_\mu(\mathrm{e}^{-2\kappa\xi}\hat{F}^{\mu\nu}) = 0 \tag{A.3}$$

and is, therefore, identical to the above equation (A.2). Equation (78) may be written like

$$u_\nu\partial_\mu(\mathrm{e}^{-2\kappa\xi}\hat{F}^{\mu\nu}) = 0 \tag{A.4}$$

together with its complex conjugate, and is, therefore, just the projection of equation (A.2) into the directions of u_μ and $\bar{u}_\mu$. Obviously, equations (A.2) are completely equivalent to equations (A.3) and (A.4).

Finally, let us discuss what happens when we further restrict to (8), that is, we set $C_\mu = 0$, too. This produces a small complication that is related to the fact that this latter restriction is no longer gauge invariant. Inserting this restriction into the Yang–Mills equation (A.1) simply gives

$$n^a\partial_\mu(\mathrm{e}^{-2\kappa\xi}H^{\mu\nu}) = 0 \tag{A.5}$$

whereas variation of the Lagrangian (13) w.r.t. $\bar{u}$ gives, in complete analogy with (A.4)

$$u_\nu\partial_\mu(\mathrm{e}^{-2\kappa\xi}H^{\mu\nu}) = 0. \tag{A.6}$$

Together with its complex conjugate, this seems to provide only two equations (the projections into the directions of u_μ and $\bar{u}_\mu$) compared to the four equations (A.5). This apparent mismatch is, however, easily understood and is related to the fact that the restriction to $C_\mu = 0$ is no longer gauge invariant. The original Yang–Mills equation is gauge covariant and leads, after inserting the restricted decomposition, to the gauge covariant field equations (A.5). Because of the abelian character of the gauge field, these equations are in fact gauge invariant after omission of the irrelevant factor n^a. On the other hand, setting $C_\mu = 0$ already in the Lagrangian involves a gauge choice, and the resulting field equations (A.6) hold only in this gauge. A gauge variation of these field equations automatically switches on the two missing components. This may be seen especially easily when the Lagrangian of the submodel is varied w.r.t. n^a instead of u, because n^a has a simple behavior under gauge transformations. Variation of the Lagrangian (9) w.r.t. n^a leads to

$$\epsilon^{abc}n^b_\mu n^c\partial_\lambda(\mathrm{e}^{-2\kappa\xi}H^{\mu\lambda}) = 0 \tag{A.7}$$

which is identical to equation (A.6) when expressed in terms of n^a instead of u. Again, equation (A.7) only consists of two components. Further, equation (A.7) consists of the gauge invariant factor

$$\partial_\lambda(\mathrm{e}^{-2\kappa\xi}H^{\mu\lambda}) \tag{A.8}$$

and the gauge dependent prefactor

$$\epsilon^{abc}n^b_\mu n^c. \tag{A.9}$$

Under a gauge variation $\delta n^a = \epsilon^{abc}n^b\alpha^c$ this factor changes according to

$$\delta\left(\epsilon^{abc}n^b_\mu n^c\right) = (\delta^{ab} - n^a n^b)\alpha^b_\mu + \left(n^a n^b_\mu - n^a_\mu n^b\right)\alpha^b. \tag{A.10}$$

As the gauge variation $\alpha^a(x)$ is completely arbitrary, already the first term proportional to α^a_μ contains, in general, projections onto the two missing components of the Yang–Mills field equations.

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